

# Some properties of $n$ -party entanglement under LOCC operations

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December 2011

## Abstract

Nielsen [4] characterized in full those 2-party quantum protocols of local operations and classical communication that transform, with probability one, a pure global initial state into a pure global final state. The present work considers the generalization of Nielsen's characterization to  $n$ -party protocols. It presents a sweeping generalization of the *only if* part of Nielsen's result. The result presented here pertains also to protocols that do not generate a final state for sure, it considers arbitrary mixed initial states instead of pure states and  $n$ -party protocols for arbitrary  $n$ 's. In this very general setting, local operations and classical communication can never decrease the expected spectra of the local mixed states in the majorization ordering. In other terms, the local states can only become purer (weakly) in expectation. The proof also provides an improvement on Nielsen's. The *if* part of Nielsen's characterization does not generalize. This is shown by studying the entanglement of three qubits. It is shown that one can find pure states of a system of three qubits that are not equivalent under unitary local operations but define local mixed states on all subparts of the system that have the same spectra. Neither equivalence of pure states under local unitary operations or accessibility under LOCC operations among a system of three qubits can be characterized by properties of the spectra of the local mixed states.

## 1 Introduction and previous work

We assume each of  $n$  parties, i.e., agents, has some piece of a quantic system. The pieces do not have to be similar. Let  $\mathbf{H} = \mathbf{H}_1 \otimes \mathbf{H}_2 \otimes \dots \otimes \mathbf{H}_n$  be a tensor product of  $n$  finite-dimensional Hilbert spaces. We consider that the global system represented by  $\mathbf{H}$  is made of  $n$  different parts, represented by  $\mathbf{H}_i$ , for  $i = 1, \dots, n$ , the  $i$ 's part being controlled by agent  $i$ . In accordance with tradition, we assume agent 1 is Alice. The different agents may be far away from one

another. From agent  $i$ 's point of view, the system it controls is represented by a mixed state of  $\mathbf{H}_i$  that depends on the global state of the system. Different actions of the agents can modify the global state and therefore the local states. This phenomenon is used to realize quantum protocols that may achieve a kind of cooperation between the different agents that is not attainable by classical means, as first showed by J. S. Bell in [1] and described in [6], Chapter 4.

We focus here on the case the possible actions of the agents are classical communication, local unitary operations and local generalized measurements. Any protocol of such actions transforms, probabilistically, the initial (mixed) state into a final (mixed) state. A particularly interesting case of such protocols is that of protocols in which one final state (pure or mixed) is obtained with probability one.

In [4], M. Nielsen characterized the final states obtainable from a given initial *pure* state with probability one by local unitary transformations, local generalized measurements and classical communication in the 2-party case ( $n = 2$ ). He showed that an initial state  $|\phi\rangle$  can be transformed into a final state  $|\psi\rangle$  in such a way iff the spectrum of the local mixed state of Alice induced by  $|\psi\rangle$  majorizes the one induced by  $|\phi\rangle$ .

The 2-party case has a distinguishing property: the spectra of the local mixed states of the two agents are closely related: they are the same up to, perhaps, some zeros, or, in other terms, their strictly positive parts are the same. This paper generalizes the *only if* part of Nielsen's result to an  $n$ -party situation and to initial states that are mixed in Section 2.7. Section 3.2 and later are devoted to showing that the generalization of the *if* part of Nielsen's result fails quite spectacularly. Even in an entangled system of three qubits one cannot decide the equivalence of pure states under local unitary operations by examining only the spectra of the mixed states of the different parts of the system.

## 2 Positive results

In this section a sweeping generalization of Nielsen's [4] result is proposed.

- Instead of considering only protocols that end up in a final global state for sure, this paper considers any LOCC protocol and the probability distribution on final global states that it generates. The notion of the expected spectrum of local states is the main tool that will enable us to study such protocols in general.
- Instead of considering only pure states as initial and final global states, this paper considers probability distributions over mixed states.
- Instead of considering 2-party entanglement, this paper considers arbitrary  $n$ -party entanglement.

## 2.1 Local states defined by a global state and their properties

### 2.1.1 Local states

Some notation will be useful. We shall always use  $i$  to represent one of the  $n$  agents. We shall use  $i^-$  to represent the set of  $n - 1$  agents that contains all agents except agent  $i$ . In the same spirit,  $\mathbf{H}_{i^-}$  represents the tensor product of all  $\mathbf{H}_j$ s, except  $\mathbf{H}_i$ . In other terms  $\mathbf{H}_{i^-} = \mathbf{H}_1 \otimes \dots \otimes \mathbf{H}_{i-1} \otimes \mathbf{H}_{i+1} \otimes \dots \otimes \mathbf{H}_n$ . Note that, for every  $i$ , we have  $\mathbf{H} = \mathbf{H}_i \otimes \mathbf{H}_{i^-}$ .

Let us assume that the global state is some mixed state, i.e., a linear, self-adjoint, weakly positive operator  $\sigma : \mathbf{H} \rightarrow \mathbf{H}$  of trace 1. Agent  $i$ , who sees only the  $\mathbf{H}_i$  part of the system, sees his system as a mixed state  $\rho_i^\sigma : \mathbf{H}_i \rightarrow \mathbf{H}_i$  defined as:

$$\rho_i^\sigma = \text{Tr}_{\mathbf{H}_{i^-}}(\sigma). \quad (1)$$

### 2.1.2 Properties of the partial trace operator

The partial trace operator satisfies the following properties, for any linear operator  $f : A \otimes B \rightarrow A \otimes B$ :

1. for any basis  $b_i$ ,  $i = 1, \dots, n$  of  $B$  and for any vectors  $x, y \in A$ ,

$$\langle x | \text{Tr}_B(f) | y \rangle = \sum_{i=1}^n \langle x \otimes b_i | f | y \otimes b_i \rangle. \quad (2)$$

2. the partial trace of the adjoint of an operator is the adjoint of the partial trace:

$$\text{Tr}_B(f^*) = (\text{Tr}_B(f))^* \quad (3)$$

and therefore the partial trace of a self-adjoint operator is self-adjoint,

3. the partial trace of a weakly positive operator is weakly positive,
4. the trace of a partial trace is the trace of the original operator

$$\text{Tr}(\text{Tr}_B(f)) = \text{Tr}(f) \quad (4)$$

5. the partial trace of the identity is multiplication by the dimension of the space,  $B$ , on which the partial trace is taken:

$$\text{Tr}_B(\text{id}_{A \otimes B}) = \dim(B) \text{id}_A, \quad (5)$$

6. for any linear operator  $g : B \rightarrow B$

$$\text{Tr}_B(f \circ (\text{id}_A \otimes g)) = \text{Tr}_B((\text{id}_A \otimes g) \circ f), \quad (6)$$

7. for any linear operator  $g : A \rightarrow A$

$$Tr_B(f \circ (g \otimes id_B)) = Tr_B(f) \circ g, \quad (7)$$

and

$$Tr_B((g \otimes id_B) \circ f) = g \circ Tr_B(f). \quad (8)$$

The case  $f$  is the projection  $P_\phi$  of  $A \otimes B$  on a unit vector  $|\phi\rangle \in A \otimes B$  is an important special case. In this case, the global state is a *pure* state, i.e., a one-dimensional subspace of  $A \otimes B$ .

A consequence of the properties of partial trace described above that will be used in Theorems 2, 3 and 12 will now be proved.

**Theorem 1** *Let  $\sigma$  be a mixed state of  $A \otimes B$  and let  $u_A : A \rightarrow A$  and  $u_B : B \rightarrow B$  be unitary maps. Then,*

$$Tr_B((u_A \otimes u_B) \circ \sigma \circ (u_A^* \otimes u_B^*)) = u_A \circ Tr_B(\sigma) \circ u_A^*.$$

**Proof:**

$$Tr_B((u_A \otimes u_B) \circ \sigma \circ (u_A^* \otimes u_B^*)) = Tr_B((u_A \otimes u_B) \circ \sigma \circ (u_A^* \otimes id_B) \circ (id_A \otimes u_B^*)) =$$

by Equation (6)

$$Tr_B((id_A \otimes u_B^*) \circ (u_A \otimes u_B) \circ \sigma \circ (u_A^* \otimes id_B)) = Tr_B((u_A \otimes id_B) \circ \sigma \circ (u_A^* \otimes id_B)) =$$

by Equations (7) and (8)

$$u_A \circ Tr_B(\sigma) \circ u_A^*.$$

■

The partial trace operator also satisfies the following. For any linear operator  $f : A \otimes B \otimes C \rightarrow A \otimes B \otimes C$ :

$$Tr_B(Tr_C(f)) = Tr_{B \otimes C}(f). \quad (9)$$

In general,  $Tr_B(g \circ f) \neq Tr_B(g) \circ Tr_B(f)$ .

### 2.1.3 Operating on mixed states

In Section 2.3 we shall discuss some operations agents can perform that transform the global state of a system. We shall now reflect on how transformations of the global state should be modeled. Typically such a transformation is modeled by some linear operator  $f : \mathbf{H} \rightarrow \mathbf{H}$ : any unit vector  $x$  of  $\mathbf{H}$  is transformed by, first, applying  $f$  to it and, then, renormalizing. If the global state is a mixed state, then, the transformation corresponding to  $f$  is the transformation that transforms the projection on  $x$ :  $P_x = |x\rangle\langle x|$  into the projection on  $f(x)$ :  $P_{f(x)} = |f(x)\rangle\langle f(x)| = f |x\rangle\langle x| f^* = f \circ P_x \circ f^*$ . The transform (by  $f$ ) of a mixed state  $\rho$  is therefore  $f \circ \rho \circ f^*$  after renormalization:

$$\rho' = \frac{f \circ \rho \circ f^*}{Tr(f \circ \rho \circ f^*)} \quad (10)$$

One easily sees that, if  $Tr(f \circ \rho \circ f^*) \neq 0$ ,  $\rho'$  is indeed a self-adjoint, weakly positive operator of trace 1. If  $Tr(f \circ \rho \circ f^*) = 0$ , the state  $\rho$  cannot be transformed by  $f$ .

## 2.2 Spectrum and majorization

The spectrum, i.e., the set of eigenvalues (with their multiplicity) of mixed local states and the majorization relation between such spectra will prove to be of cardinal importance in our study of transformations of local states brought about by the different agents activities. Since mixed states are self-adjoint, weakly positive operators of trace one, their spectrum is composed of real nonnegative numbers whose sum is equal to one. **We shall always order such spectra in decreasing order.** If  $Sp(\rho) = (\lambda_1, \dots, \lambda_n)$ ,  $\lambda_1$  is the largest eigenvalue of  $\rho$  and  $\lambda_n$  the smallest. This enables us to define sums and convex combinations of spectra: the first element of  $1/2 Sp(\rho) + 1/2 Sp(\sigma)$ , for example, is the average of the largest eigenvalues of  $\rho$  and  $\sigma$ . When we add spectra of different lengths, or compare them as below, we always padd the shorter spectrum with zeros on the right.

**Definition 1** Let  $\lambda = (\lambda_1, \dots, \lambda_n)$ , and  $\mu = (\mu_1, \dots, \mu_n)$  be spectra (suitably padded). We say that  $\lambda$  majorizes  $\mu$  and write  $\lambda \succeq \mu$  iff for every  $k = 1, \dots, n$ , one has:

$$\sum_{j=1}^k \lambda_j \geq \sum_{j=1}^k \mu_j. \quad (11)$$

Let  $\rho : A \rightarrow A$  and  $\sigma : B \rightarrow B$  be any two mixed states, we say that  $\rho$  majorizes  $\sigma$  and write  $\rho \succeq \sigma$  iff  $Sp(\rho) \succeq Sp(\sigma)$ .

Note that a pure state majorizes any state. The majorization relation is a pre-order: reflexive and transitive. Two mixed states are equivalent in the majorization order iff they have the same spectrum. The reader should think of Definition 1 in the context of local mixed states  $\rho$  and  $\sigma$ . We now want to define a relation on global states.

**Definition 2** Let  $\rho, \sigma : \mathbf{H} \rightarrow \mathbf{H}$  be global mixed states, where  $\mathbf{H} = \mathbf{H}_1 \otimes \dots \otimes \mathbf{H}_n$ . We shall say that  $\rho$  is stronger than  $\sigma$  iff, for every  $i = 1, \dots, n$ , one has:

$$Tr_{\mathbf{H}_{i-}}(\rho) \succeq Tr_{\mathbf{H}_{i-}}(\sigma). \quad (12)$$

The tool we shall use in Section 2.5.3 to prove majorization properties is Corollary 1 below. Its proof is based on Theorem 15 found in Appendix A with a proof.

The following is found in [3] p. 241.

**Corollary 1** For any self-adjoint matrices  $A$  and  $B$ ,  $A, B : \mathbf{H} \rightarrow \mathbf{H}$ , one has  $Sp(A) + Sp(B) \succeq Sp(A + B)$ .

**Proof:** Let  $Sp(A + B) = \{\nu_i\}_{i=1, \dots, n}$ . For any  $q$ ,  $1 \leq q \leq n$ , by Theorem 15, the sum  $\sum_{i=1}^q \nu_i$  is the maximum value taken by  $w_{A+B}(b)$  on all  $q$ -bases for  $\mathbf{H}$ . But  $w_{A+B}(b) = w_A(b) + w_B(b)$ ,  $w_A(b) \leq \sum_{i=1}^q \lambda_i$  and  $w_B(b) \leq \sum_{i=1}^q \mu_i$ . ■

## 2.3 Local operations and classical communication

### 2.3.1 Local operations

It is time to have a closer look at those operations that different agents that share some entangled system can perform in the kind of protocols studied in the theory of quantum information. Such operations are traditionally described as LOCC: local operations and classical communication.

From now on, we shall assume that the first agent, the one that sees the  $\mathbf{H}_1$  part of the system is called Alice and we shall assume that Alice is the only active agent. Obviously, what we say about Alice's actions applies to any other agent's actions. The characteristic feature of a local operation of Alice is that it acts only on  $\mathbf{H}_1$ . Any local operation of Alice can be characterized by a linear operator  $f : \mathbf{H}_1 \rightarrow \mathbf{H}_1$ . Its effect on the global state  $\rho$  is to transform the global state  $\rho$  into  $f' \circ \rho$  where  $f'$  is defined below.

**Definition 3** For any  $f : \mathbf{H}_1 \rightarrow \mathbf{H}_1$  we let  $f' = f \otimes id_{\mathbf{H}_{1-}}$ .

There are two kinds of local operations that Alice can perform:

- unitary transformations and
- measurements.

We shall study them in Sections 2.4 and 2.5 respectively.

### 2.3.2 Classical communication

The second sort of action that Alice can take is to transfer information, in other terms *talk*, to some of the other agents. This is done by classical means and therefore does not change the global state. This operation is particularly important in connection with measurements. Typically, Alice will communicate to her partners the results of generalized measurements she has performed. Once this is done, the ensuing operations of the other agents may depend on the information they received from Alice on the results of her measurements.

## 2.4 Unitary local operations

If  $U : \mathbf{H}_1 \rightarrow \mathbf{H}_1$  is unitary, then  $U' = U \otimes id_{\mathbf{H}_{1-}} : \mathbf{H} \rightarrow \mathbf{H}$  is also unitary. Therefore  $Tr(U' \circ \rho \circ U'^*) = Tr(\rho \circ U'^* \circ U') = Tr(\rho) = 1$  and the normalization factor is 1. The mixed state  $\rho$  is transformed into  $\rho' = U' \circ \rho \circ U'^*$ , by Equation (10).

We can now describe in full the effect of Alice's local unitary operation on all the local mixed states

**Theorem 2** *Alice's local unitary operations do not change the local mixed states of other agents.*

**Proof:** For  $i > 1$

$$\rho_i^{U' \circ \sigma \circ U'^*} = \text{Tr}_{\mathbf{H}_{i-}}((U \otimes \text{id}_{\mathbf{H}_{i-}}) \circ \sigma \circ (U^* \otimes \text{id}_{\mathbf{H}_{i-}})) = \text{Tr}_{\mathbf{H}_{i-}}(\sigma) = \rho_i^\sigma \quad (13)$$

by Theorem 1.  $\blacksquare$

This is indeed as expected: Bob is not affected by and cannot detect a unitary operation of Alice. The effect of a unitary operation performed by Alice on her own local state is also as expected.

**Theorem 3** *A unitary local operation  $U$  of Alice transforms her local mixed state  $\rho_1^\sigma$  into  $U \circ \rho_1^\sigma \circ U^*$ , as if Alice were alone. Therefore it does not change the spectrum of her local state.*

**Proof:**

$$\rho_1^{U' \circ \sigma \circ U'^*} = \text{Tr}_{\mathbf{H}_{1-}}((U \otimes \text{id}_{\mathbf{H}_{1-}}) \circ \sigma \circ (U^* \otimes \text{id}_{\mathbf{H}_{1-}})) = U \circ \rho_1^\sigma \circ U^* \quad (14)$$

by Theorem 1. Note, now, that for any operator  $f$ ,  $Sp(U \circ f \circ U^*) = Sp(f)$ .  $\blacksquare$

## 2.5 Local measurements

### 2.5.1 Generalized measurements

If one decides to measure some observable represented by a generalized measurement, i.e., a sequence  $f_1, \dots, f_m$  of operators:  $f_j : A \rightarrow A$  for  $j = 1, \dots, m$  that satisfy:

$$\sum_{j=1}^m f_j^* \circ f_j = \text{id}_A \quad (15)$$

one will obtain some result, i.e., some  $j$  for his measurement. The state of the system defines only a probability distribution on the possible results. If the state is  $\rho$  then the probability of obtaining result  $j$  is given by:

$$p_j = \text{Tr}(f_j \circ \rho \circ f_j^*) = \text{Tr}(f_j \circ f_j^* \circ \rho). \quad (16)$$

The state  $\rho$  is changed by the measurement. If  $p_j = 0$ , the result  $j$  is never obtained. If  $p_j > 0$ , the new state is given by:

$$\rho'_j = \frac{1}{p_j} f_j \circ \rho \circ f_j^*. \quad (17)$$

A word of caution is in order here. The way our generalized measurements have been described above corresponds to what is usually named *measuring* POVM (positive operator valued measure) measurement in the literature, in which the agent records the result of the measurement performed. Another form of generalized measurement has also been considered in the literature: *trace preserving* POVM measurements, in which the agent does not record the result of the measurement. The notion of LOCC operations considered in this paper does not include trace preserving operations. For a description of these different types of measurement, see, for example, the very useful survey [5].

### 2.5.2 Expected spectrum

Measurements, contrary to the unitary and classical operations considered in Sections 2.4 and 2.3.2 respectively, do not transform a state into a state. It is a fundamental property of Quantum Physics that measurements transform a state (pure or mixed) into a probability distribution on states (pure or mixed). For many purposes, in Quantum Physics, a probability distribution on states can be confused with the (mixed) state which is the linear combination of the original states weighted by their respective probabilities. Such a confusion causes no problem as long as the quantic operations considered are linear. But measurements are not linear transformations and, as noticed in Section 2.2, we wish to attach a spectrum with such a probability distribution. The spectrum associated with a system that is in state  $\rho_i$  with probability  $p_i$ ,  $i = 1, 2$ , is  $\sum_{i=1}^2 p_i Sp(\rho_i)$ , i.e., the spectrum the  $k$ 'th element of which is a combination of the  $k$ 'th elements of  $\rho_1$  and  $\rho_2$  respectively. This spectrum is not equal to the spectrum of the state  $\sum_{i=1}^2 p_i \rho_i$ . In the sequel, probability distributions on states are not to be confused with (mixed) states. Section 2.5.3 studies the expected spectrum that results from a local generalized measurement.

### 2.5.3 Local generalized measurements

Suppose Alice performs a local generalized measurement  $f_1, \dots, f_m : \mathbf{H}_1 \rightarrow \mathbf{H}_1$  on her piece of the global system, which is in global state  $\sigma$ . If, following our custom, we let  $f'_i : f_i \otimes id_{\mathbf{H}_{i-}}$ , the probability of obtaining result  $j$  is given by:

$$\begin{aligned} p_j &= Tr(f'_j \circ f'^*_j \circ \sigma) = Tr(Tr_{\mathbf{H}_{1-}}(f'_j \circ f'^*_j \circ \sigma)) = \\ &Tr(f_j \circ f_j^* \circ Tr_{\mathbf{H}_{1-}}(\sigma)) = Tr(f_j \circ f_j^* \circ \rho_1^\sigma) \end{aligned} \quad (18)$$

by Equations (16), (4), (8) and (1). We note that, as expected,

$$\begin{aligned} \sum_{j=1}^m p_j &= \sum_{j=1}^m Tr(f_j \circ f_j^* \circ \rho_1^\sigma) = Tr\left(\sum_{j=1}^m f_j \circ f_j^* \circ \rho_1^\sigma\right) = \\ &Tr\left(\left(\sum_{j=1}^m f_j \circ f_j^*\right) \rho_1^\sigma\right) = Tr(\rho_1^\sigma) = 1. \end{aligned} \quad (19)$$

Alice may obtain result  $j$  for her measurement only if  $p_j > 0$ , and, then, the new global state of the system is:

$$\sigma'_j = \frac{1}{p_j} f'_j \circ \sigma \circ f'^*_j. \quad (20)$$

If  $p_j = 0$ , then the operator  $Tr_{\mathbf{H}_{i-}}(f'_j \circ \sigma \circ f'^*_j)$  is self-adjoint, weakly positive and has a trace equal to 0, it is therefore equal to zero. We conclude that, for any  $j = 1, \dots, m$ , one has:

$$p_j \sigma'_j = f'_j \circ \sigma \circ f'^*_j. \quad (21)$$



We shall now study the expected effect of a measurement of Alice on each of the local states. We shall show that the expected local spectrum always majorizes the current local spectrum. The proof is based on Corollary 1, but we need to distinguish two cases. First, we treat the effect of Alice's measurement on other agents' local states, and then its effect on Alice's own local state.

#### 2.5.4 Effect of Alice's measurements on other agents' state

The new local state of any agent  $i$ , other than Alice, i.e.,  $i > 1$ , after Alice has obtained result  $j$  is therefore:

$$\rho_i^{\sigma'_j} = \text{Tr}_{\mathbf{H}_{i-}}(\sigma'_j) = \frac{1}{p_j} \text{Tr}_{\mathbf{H}_{i-}}(f'_j \circ \sigma \circ f'^*_j) \quad (22)$$

by Equation (6).

An example of *two* entangled systems presented in Appendix B shows that it is not the case, even for 2-entanglement, that the eigenvalues of  $\rho_i^{\sigma'_j}$  majorize those of  $\rho_i^\sigma$ . An individual measurement by Alice can make the mixed state of another agent more chaotic.

Nevertheless, we shall show that, in expectation, taking into account all the possible results of Alice's measurement, the eigenvalues of the new mixed state of any agent different from Alice majorize those of the old state of this agent. The mixed state of the agent becomes less chaotic, *purier*.

**Theorem 4** *The expected eigenvalues of Bob's local state after a measurement by Alice majorize the eigenvalues of Bob's current local state.*

$$\sum_{j=1}^m p_j \text{Sp}(\rho_i^{\sigma'_j}) \succeq \text{Sp}(\rho_i^\sigma) \quad (23)$$

for any  $i > 1$ .

**Proof:** By Equation (22) we have:

$$\sum_{j=1}^m p_j \text{Sp}(\rho_i^{\sigma'_j}) = \sum_{j=1}^m \text{Sp}(p_j \rho_i^{\sigma'_j}) = \sum_{j=1}^m \text{Sp}(\text{Tr}_{\mathbf{H}_{i-}}(f'_j \circ \sigma \circ f'^*_j)) \quad (24)$$

and by Corollary 1:

$$\sum_{j=1}^m \text{Sp}(\text{Tr}_{\mathbf{H}_{i-}}(f'_j \circ \sigma \circ f'^*_j)) \succeq \text{Sp}\left(\sum_{j=1}^m \text{Tr}_{\mathbf{H}_{i-}}(f'_j \circ \sigma \circ f'^*_j)\right). \quad (25)$$

By property 6 of the partial trace operator in Section 2.1:

$$\text{Tr}_{\mathbf{H}_{i-}}(f'_j \circ \sigma \circ f'^*_j) = \text{Tr}_{\mathbf{H}_{i-}}(f'^*_j \circ f'_j \circ \sigma)$$

and, by Equation (15)

$$\sum_{j=1}^m \text{Tr}_{\mathbf{H}_{i-}}(f_j'^* \circ f_j' \circ \sigma) = \text{Tr}_{\mathbf{H}_{i-}}\left(\sum_{j=1}^m f_j'^* \circ f_j' \circ \sigma\right) = \text{Tr}_{\mathbf{H}_{i-}}(\sigma) = \rho_i^\sigma. \quad (26)$$

■

### 2.5.5 Effect of Alice's measurement on her own local state

Once Alice has obtained result  $j$  her new local state, using Equations (7) and (8) is given by:

$$\rho_1^{\sigma'_j} = \text{Tr}_{\mathbf{H}_{1-}}(\sigma'_j) = \frac{1}{p_j} \text{Tr}_{\mathbf{H}_{1-}}(f_j' \circ \sigma \circ f_j'^*) = \quad (27)$$

$$\frac{1}{p_j} f_j \circ \text{Tr}_{\mathbf{H}_{1-}}(\sigma) \circ f_j^* = \frac{1}{p_j} f_j \circ \rho_1^\sigma \circ f_j^*.$$

Indeed, Alice's state can be computed locally, using the local mixed state  $\rho_1^\sigma$  and the local operations  $f_j$ . We may now prove a result similar to Theorem 4.

**Theorem 5** *The expected eigenvalues of Alice's local state after her measurement majorize the eigenvalues of her current local state.*

$$\sum_{j=1}^m p_j \text{Sp}(\rho_1^{\sigma'_j}) \succeq \text{Sp}(\rho_1^\sigma). \quad (28)$$

**Proof:** By Equation (27)

$$\sum_{j=1}^m p_j \text{Sp}(\rho_1^{\sigma'_j}) = \sum_{j=1}^m \text{Sp}(p_j \rho_1^{\sigma'_j}) = \sum_{j=1}^m \text{Sp}(f_j \circ \rho_1^\sigma \circ f_j^*) \quad (29)$$

The operator  $\rho_1^\sigma$  is self-adjoint and weakly positive, it has therefore a square root, i.e., a self-adjoint, weakly positive operator  $\alpha : \mathbf{H}_1 \rightarrow \mathbf{H}_1$  such that  $\rho_1^\sigma = \alpha \circ \alpha^*$ . Let  $\beta_j = f_j \circ \alpha$ . We have  $f_j \circ \rho_1^\sigma \circ f_j^* = \beta_j \circ \beta_j^*$ . But  $\text{Sp}(\beta_j \circ \beta_j^*) = \text{Sp}(\beta_j^* \circ \beta_j)$ , as is proved in Theorem 16 in Appendix C. We conclude that

$$\text{Sp}(f_j \circ \rho_1^\sigma \circ f_j^*) = \text{Sp}(\alpha^* \circ f_j^* \circ f_j \circ \alpha).$$

By Corollary 1 and Equation (15) one has:

$$\sum_{j=1}^m \text{Sp}(\alpha^* \circ f_j^* \circ f_j \circ \alpha) \succeq \text{Sp}\left(\sum_{j=1}^m \alpha^* \circ f_j^* \circ f_j \circ \alpha\right) = \text{Sp}(\alpha^* \circ \alpha) = \text{Sp}(\rho_1^\sigma). \quad (30)$$

■

Theorem 5 also holds when Alice is alone in the universe: in expectation the result of a generalized measurement always majorizes the initial state. As a consequence, in expectation, the entropy cannot increase as a result of a measurement. This fits in well with the idea that a measurement always reduces uncertainty.

## 2.6 LOCC operations weakly increase the spectra of all local states in the majorization order

**Theorem 6** *In any LOCC protocol, the spectrum of any initial local state is majorized by its expected final local spectrum in the majorization order.*

**Proof:** By Section 2.3.2, Theorems 2, 3, 4 and 5 no step in the protocol can decrease any local spectrum in the majorization order. ■

## 2.7 Derivation of a generalization of one-half of Nielsen's result

We can now derive a generalization of one half (the only if part) of Nielsen's Theorem 1 in [4].

**Corollary 2** *If there is an  $n$ -party protocol consisting of local unitary operations, local generalized measurements and classical communication that, starting in a mixed global state  $\sigma$  terminates for sure, i.e., with probability one, in mixed global state  $\sigma'$ , then  $\sigma'$  is stronger than  $\sigma$  in the sense of Definition 2.*

**Proof:** At each step of the protocol, we have shown that, for any agent, the initial mixed local state is majorized by the expected spectrum of the final mixed local state. If the final global state is, for sure,  $\sigma'$ , the final mixed local states are  $\rho_i^{\sigma'}$  and the expected spectra are  $Sp(\rho_i^{\sigma'})$ . We conclude that, for every  $i$ ,  $1 \leq i \leq n$  one has:  $\rho_i^{\sigma'} \succeq \rho_i^{\sigma}$ , proving our claim. ■

One may note that our results do not use Schmidt's decomposition, which is used heavily in [4].

## 2.8 Generalization to an arbitrary probability $p$

We generalize the *only if* part of Nielsen's result (Corollary 2) to the case the LOCC protocol ends in state  $\sigma'$  only with a probability  $p$  that may be less than unity.

First, we define a relation of approximate majorization that generalizes Definitions 1 and 2.

**Definition 4** *Let  $c$  be a real number. Let  $\lambda = (\lambda_1, \dots, \lambda_n)$ , and  $\mu = (\mu_1, \dots, \mu_n)$  be spectra (suitably padded). We say that  $\lambda$   $c$ -majorizes  $\mu$  and write  $\lambda \succeq_c \mu$  iff for every  $k = 1, \dots, n$ , one has:*

$$\sum_{j=1}^k \lambda_j \geq \sum_{j=1}^k \mu_j - 1 + c. \quad (31)$$

*Let  $\rho : A \rightarrow A$  and  $\sigma : B \rightarrow B$  be any two mixed states, we say that  $\rho$   $c$ -majorizes  $\sigma$  and write  $\rho \succeq_c \sigma$  iff  $Sp(\rho) \succeq_c Sp(\sigma)$ . If  $\rho$  and  $\sigma$  are global mixed states, as in*

Definition 2, we say that  $\rho$  is  $c$ -stronger than  $\sigma$  iff, for every  $i = 1, \dots, n$ , one has:

$$\text{Tr}_{\mathbf{H}_{i-}}(\rho) \succeq_c \text{Tr}_{\mathbf{H}_{i-}}(\sigma). \quad (32)$$

The relation  $\succeq_c$  is not, in general, transitive but it obviously satisfies the following.

- Theorem 7** 1. For any real numbers  $c$  and  $d$ , such that  $c > d$ , for any  $\sigma$  and  $\tau$ , if  $\sigma \succeq_c \tau$ , then  $\sigma \succeq_d \tau$ ,
2. for any  $c > 1$ , for no  $\sigma$  and  $\tau$  do we have  $\sigma \succeq_c \tau$ ,
3. the relation  $\succeq_1$  is the majorization relation  $\succeq$ ,
4. for any  $c \leq 0$ , for any  $\sigma, \tau$  we have  $\sigma \succeq_c \tau$ .

**Theorem 8** If there is an  $n$ -party protocol consisting of local unitary operations, local generalized measurements and classical communication that, starting in a mixed global state  $\sigma$  terminates with probability at least  $p$ , in mixed global state  $\sigma'$ , then  $\sigma'$  is  $p$ -stronger than  $\sigma$ .

**Proof:** Suppose  $q \geq p$  is the probability with which state  $\sigma'$  is attained. By Theorem 6 we have  $q \text{Sp}(\text{Tr}_{i-}(\sigma')) + (1 - q) S \succeq \text{Sp}(\text{Tr}_{i-}(\sigma))$ , where  $S$  is the spectrum expected if the protocol does not attain  $\sigma'$ . But  $q \leq 1$  and  $1 - q \leq 1 - p$  and we have:  $\text{Sp}(\text{Tr}_{i-}(\sigma')) + (1 - p) S \succeq \text{Sp}(\text{Tr}_{i-}(\sigma))$ . Denote the eigenvalues of  $\text{Tr}_{i-}(\sigma)$  and  $\text{Tr}_{i-}(\sigma')$  by  $\lambda_i$  and  $\lambda'_i$  respectively and by  $\mu_i$  those of  $S$ . For any  $k \leq n$ , we have:  $\sum_{i=1}^k \lambda'_i + (1 - p) \sum_{i=1}^k \mu_i \geq \sum_{i=1}^k \lambda_i$  and therefore  $\sum_{i=1}^k \lambda'_i + 1 - p \geq \sum_{i=1}^k \lambda_i$ . ■

### 3 Negative results

In this section, we describe a number of ways certain generalizations of *if* part of Nielsen's result fail.

#### 3.1 Mixed states

The converse of Corollary 2 does not hold. In fact, it fails quite dramatically. Consider any mixed global state  $\sigma$  and the mixed global state  $\sigma' = \rho_1^\sigma \otimes \dots \otimes \rho_n^\sigma$ . It is clear that, for every  $i$ ,  $1 \leq i \leq n$ ,  $\rho_i^{\sigma'} = \rho_i^\sigma$  and therefore  $\sigma'$  and  $\sigma$  are equivalently strong. But, there is a protocol transforming, for sure,  $\sigma'$  into  $\sigma$  only if  $\sigma$  is itself a product state since product states, such as  $\sigma'$ , are transformed into product states by local operations. In general, therefore, there is no protocol transforming  $\sigma'$  into  $\sigma$ .

Any LOCC protocol transforms a pure state, or a distribution over such states, into a distribution over pure states. One possible generalization of Nielsen's result may be to ask whether, given two pure global states  $s$  and  $s'$  such that  $s'$  is stronger than  $s$  there is always a transformation of  $s$  into  $s'$ ,

for sure, by local operations and classical communication. Nielsen has shown that, for 2-entanglement, this is the case. Theorem 14 will show, by studying systems of three qubits, that, for  $n$ -entanglement with  $n > 2$ , this is not the case.

Another angle of attack may be to try and generalize Nielsen's result, for  $n = 2$  to mixed states. A serious problem is already apparent when one studies equivalence of mixed global states under local unitary operations for 2-entanglement. Let  $\sigma, \tau$  be mixed states of  $A \otimes B$ . If there are unitary local operations  $u_A : A \rightarrow A$  and  $u_B : B \rightarrow B$  such that  $\tau = (u_A \otimes u_B) \circ \sigma \circ (u_A^* \otimes u_B^*)$  then, necessarily, the states  $\sigma$  and  $\tau$  have the same spectrum, the local states  $Tr_B(\sigma)$  and  $Tr_B(\tau)$  have the same spectrum and so do  $Tr_A(\sigma)$  and  $Tr_A(\tau)$ . Theorem 13 will show that, even for two qubits, there are such states  $\sigma$  and  $\tau$  that are not equivalent under local unitary operations.

### 3.2 Three qubits

To every unit vector  $x$  of  $A$  one associates its projection  $p_x$ , denoted  $|x\rangle\langle x|$  in Dirac's notation, which is a mixed state of  $A$ . A qubit  $\mathbf{Q}$  is a two-dimensional Hilbert space on the complex field. Let  $\mathbf{H} = \mathbf{Q}_1 \otimes \mathbf{Q}_2 \otimes \mathbf{Q}_3$  be the tensor product of three qubits. Given any unit vector  $x \in \mathbf{H}$  one defines mixed states on each of the qubits by:

$$\rho_i^x = Tr_{\mathbf{H}_{i-}}(p_x) \quad (33)$$

for  $i = 1, 2, 3$ , where  $\mathbf{H}_{1-} = \mathbf{H}_2 \otimes \mathbf{H}_3$ ,  $\mathbf{H}_{2-} = \mathbf{H}_1 \otimes \mathbf{H}_3$  and  $\mathbf{H}_{3-} = \mathbf{H}_1 \otimes \mathbf{H}_2$ . Let  $|0_i\rangle$  and  $|1_i\rangle$  be a basis for  $\mathbf{Q}_i$ , for  $i = 1, 2, 3$ . When the sub-index is obvious from the context we shall not mention it and we shall abuse notations. For example  $|010\rangle$  denotes the product state of  $\mathbf{H}$ :  $|0_1\rangle \otimes |1_2\rangle \otimes |0_3\rangle$ . In the sequel, indices  $i$  and  $j$  will range over the qubits:  $i, j \in \{1, 2, 3\}$  and  $k, l$  and  $m$  range over the indices of the base vectors:  $k, l, m \in \{0, 1\}$ . In formulas using summation over those indices we shall dispense with specifying the bounds.

### 3.3 The question

The main technical question we ask and answer is the following: given any three mixed qubit states:  $\sigma_i : \mathbf{Q}_i \rightarrow \mathbf{Q}_i$ , is there some *pure* state  $x$  of  $\mathbf{H}$  such that  $\rho_i^x = \sigma_i$  for every  $i$ ? Our answer to the question is not complete, but we shall learn enough to demonstrate that  $n$ -party entanglement for  $n > 2$  has properties very different from 2-party entanglement.

### 3.4 The equations

Without loss of generality, we let, for any  $i$ , the eigenvalues of  $\sigma_i$  be  $\lambda_i^0 \geq \lambda_i^1 \geq 0$  such that  $\sum_k \lambda_i^k = 1$ , we assume, w.l.o.g., that  $\lambda_1^0 \geq \lambda_2^0 \geq \lambda_3^0$  and we let  $|0_i\rangle$  be an eigenvector of  $\sigma_i$  for the eigenvalue  $\lambda_i^0$  and  $|1_i\rangle$  be an eigenvector of  $\sigma_i$  for the eigenvalue  $\lambda_i^1$ .

Let  $x = \sum_{k,l,m} x_{klm} |klm\rangle$  be any vector of  $\mathbf{H}$ . We are now going to write down equations (in the complex coefficients  $x_{klm}$ ) to characterize those pure states  $x$  such that  $\rho_i^x = \sigma_i$ , for any  $i$ .

A first equation requires  $x$  to be a unit vector:

$$1 = \|x\| = \sum_{k,l,m} x_{klm} x_{klm}^*. \quad (34)$$

Our next equations express that  $|0_i\rangle$  is an eigenvector of  $\sigma_i$  for eigenvalue  $\lambda_i^0$ . We have:

$$\lambda_i^0 = \langle 0_i | \sigma_i | 0_i \rangle = \sum_{k,l} x_{\tilde{i}(0kl)} x_{i(0kl)}^* \quad (35)$$

where  $\tilde{1}(klm) = klm$ ,  $\tilde{2}(klm) = mkl$ ,  $\tilde{3}(klm) = lmk$  and

$$0 = \langle 1_i | \sigma_i | 0_i \rangle = \sum_{k,l} x_{\tilde{i}(1kl)} x_{i(0kl)}^*. \quad (36)$$

Our last equations express that  $|1_i\rangle$  is an eigenvector of  $\rho_i$  for eigenvalue  $\lambda_i^1$ .

$$\lambda_i^1 = \langle 1_i | \sigma_i | 1_i \rangle = \sum_{k,l} x_{\tilde{i}(1kl)} x_{i(1kl)}^* \quad (37)$$

and

$$0 = \langle 0_i | \sigma_i | 1_i \rangle = \sum_{k,l} x_{\tilde{i}(0kl)} x_{i(1kl)}^*. \quad (38)$$

One notices that, for every  $i$ , since  $\sum_k \lambda_i^k = 1$ , one can obtain Equation (37) by subtracting Equation (35) from Equation (34) and that Equation (38) is implied, by transposition, by Equation (36). Therefore we conclude:

**Theorem 9** *The 7 Equations (34), (35) and (36) characterize those vectors  $x$  for which  $\rho_i^x = \sigma_i$  for every  $i$ .*

### 3.5 Study of solutions

We want to study solutions to those equations, but are far from a complete understanding. We shall see in Theorem 10 that the equations above do not always, i.e., for any  $\lambda_i$ 's, have a solution, but we shall describe, in Theorem 11, a 3-dimensional domain (in the  $\lambda$ 's) in which solutions always exist and a sub-domain in which multiple solutions coexist.

Our first result is a full characterization of the solutions in the special case in which the system is a tensor product of a first qubit and a system of two qubits, i.e., in the case  $\lambda_1^0 = 1$ .

**Theorem 10** *For  $\lambda_1^0 = 1$ , Equations (34), (35) and (36) have a solution iff  $\lambda_2^0 = \lambda_3^0$ , equivalently  $\lambda_1^0 + \lambda_2^0 - \lambda_3^0 \leq 1$ .*

**Proof:** Two proofs will be presented.

1. First proof: if  $\lambda_1^0 = 1$ , the global state  $x$  is a tensor product  $w \otimes y$  where  $w \in \mathbf{Q}_1$  and  $y \in \mathbf{Q}_2 \otimes \mathbf{Q}_3$ . Schmidt's decomposition of  $y$  shows that the spectra of the mixed local states defined by  $y$  are equal, i.e.,  $\lambda_2^0 = \lambda_3^0$ . Conversely, if  $\lambda_2^0 = \lambda_3^0$  the eigenvectors of the two mixed local states define the Schmidt's decomposition of a suitable  $y$ .
2. Second proof: Suppose  $x_{klm}$  is a solution. Since  $\lambda_1^0 = 1$ , Equations (34) and (35) for  $i = 1$  imply that  $x_{1lm} = 0$  for any  $l, m$ . Equations (36) for  $i = 2, 3$  can now be written as:

$$\begin{bmatrix} x_{010} & x_{001}^* \\ x_{001} & x_{010}^* \end{bmatrix} \begin{bmatrix} x_{000}^* \\ x_{011} \end{bmatrix} = 0. \quad (39)$$

We conclude that:

- either  $x_{000} = x_{011} = 0$ , in which case Equations (35) can be written
$$1 = \lambda_1^0 = x_{001}x_{001}^* + x_{010}x_{010}^*, \quad \lambda_2^0 = x_{001}x_{001}^*, \quad \lambda_3^0 = x_{010}x_{010}^*$$
and we conclude that  $\lambda_2^0 + \lambda_3^0 = 1$  and therefore  $\lambda_2^0 = \lambda_3^0 = 1/2$ ,
- or  $x_{010}x_{010}^* = x_{001}x_{001}^*$ , in which case Equations (35) for  $i = 2, 3$  now imply  $\lambda_2^0 = \lambda_3^0$ .

For the *if* part, notice that if  $\lambda_1^0 = 1$ ,  $\lambda_2^0 = \lambda_3^0$ ,  $x_{1lm} = 0$  for every  $l, m$  and  $x_{010} = x_{001} = 0$ , then the equations boil down to:

$$1 = x_{000}x_{000}^* + x_{011}x_{011}^*, \quad \lambda_2^0 = x_{000}x_{000}^* \quad (40)$$

which can be solved.

■

**Definition 5** We shall say that an index  $klm$  is odd iff the number of ones is odd, and that it is even iff the number of ones is even.

Two families of solutions will be presented, each under a condition concerning the  $\lambda$ 's. Note that the second condition  $\lambda_1^0 + \lambda_2^0 + \lambda_3^0 \leq 2$  implies the first condition  $\lambda_1^0 + \lambda_2^0 - \lambda_3^0 \leq 1$ , which holds on a larger domain of parameters.

**Theorem 11** Solutions to Equations (34), (35) and (36) for  $i = 1, 2, 3$  are provided, for any phases  $\theta_{klm} \in [0, 2\pi]$ , by:

- if  $\lambda_1^0 + \lambda_2^0 - \lambda_3^0 \leq 1$

$$y_{klm} = 0 \text{ for every odd index } klm \quad (41)$$

$$y_{000} = e^{\theta_{000}} \sqrt{2(\lambda_1^0 + \lambda_2^0 + \lambda_3^0 - 1)} / 2 \quad (42)$$

$$y_{011} = e^{\theta_{011}} \sqrt{2(\lambda_1^0 - \lambda_2^0 - \lambda_3^0 + 1)} / 2 \quad (43)$$

$$y_{101} = e^{\theta_{101}} \sqrt{2(-\lambda_1^0 + \lambda_2^0 - \lambda_3^0 + 1)} / 2 \quad (44)$$

$$y_{110} = e^{\theta_{110}} \sqrt{2(-\lambda_1^0 - \lambda_2^0 + \lambda_3^0 + 1)} / 2 \quad (45)$$

- if  $\lambda_1^0 + \lambda_2^0 + \lambda_3^0 \leq 2$

$$z_{klm} = 0 \text{ for every even index } klm \quad (46)$$

$$z_{001} = e^{\theta_{001}} \sqrt{2(\lambda_1^0 + \lambda_2^0 - \lambda_3^0)} / 2 \quad (47)$$

$$z_{010} = e^{\theta_{010}} \sqrt{2(\lambda_1^0 - \lambda_2^0 + \lambda_3^0)} / 2 \quad (48)$$

$$z_{100} = e^{\theta_{100}} \sqrt{2(-\lambda_1^0 + \lambda_2^0 + \lambda_3^0)} / 2 \quad (49)$$

$$z_{111} = e^{\theta_{111}} \sqrt{2(-\lambda_1^0 - \lambda_2^0 - \lambda_3^0 + 2)} / 2 \quad (50)$$

Notice that the quantities of which we take a square root are indeed non-negative: since  $1 \geq \lambda_1^0 \geq \lambda_2^0 \geq \lambda_3^0 \geq 1/2$  (Equations (47), (48), (49), (42), (43) and (44) and by the specific assumptions (Equations (50) and (45)).

**Proof:** By inspection. Notice that, in Equations (36), every term is the product of a variable of odd index by a variable of even index and therefore each of Equations (46) and (41) guarantees that they are satisfied. ■

Note that, if  $\lambda_1^0 = 1$ , as in Theorem 10, the first condition  $\lambda_1^0 + \lambda_2^0 + \lambda_3^0 \leq 2$  is equivalent to  $\lambda_2^0 = \lambda_3^0 = 1/2$  and that, in this case, the solution  $z$  of Theorem 11 is different from the solution  $x$  of Theorem 10. Note also that, for  $\lambda_1^0 = 1$ , the second condition  $\lambda_1^0 + \lambda_2^0 - \lambda_3^0 \leq 1$  is equivalent to  $\lambda_2^0 = \lambda_3^0$  as noticed in Theorem 10 and that, in this case, the solution  $y$  of Theorem 11 is identical to the solution  $x$  of Theorem 10. At this point it is natural to ask whether the equations above have a solution iff  $\lambda_1^0 + \lambda_2^0 - \lambda_3^0 \leq 1$ . No answer is available at the moment.

### 3.6 A property of equivalence under unitary operations

**Theorem 12** Suppose  $\sigma$  and  $\tau$  are mixed states of  $A \otimes B$  such that:

1.  $Tr_B(\tau) = Tr_B(\sigma)$ ,  $Tr_A(\tau) = Tr_A(\sigma)$  and no eigenvalue of any of those operators is degenerate
2. and there are unitary maps  $u_A : A \rightarrow A$  and  $u_B : B \rightarrow B$  such that  $\tau = (u_A \otimes u_B) \circ \sigma \circ (u_A^* \otimes u_B^*)$ .

Then, if  $\sigma_{i,j}$  and  $\tau_{i,j}$  are the elements of the matrices representing  $\sigma$  and  $\tau$  in the basis whose elements are the tensor products of eigenbases for the traces on  $A$  and  $B$ , then, for any  $i, j$  one has:  $|\tau_{i,j}| = |\sigma_{i,j}|$ .

**Proof:** Assume all assumptions of the theorem are satisfied. By Theorem 1, we have:

$$Tr_B(\sigma) = Tr_B(\tau) = Tr_B((u_A \otimes u_B) \circ \sigma \circ (u_A^* \otimes u_B^*)) = u_A \circ Tr_B(\sigma) \circ u_A^*.$$



Therefore we have:  $Tr_B(\sigma) \circ u_A = u_A \circ Tr_B(\sigma)$ . Let  $x$  be an eigenvector of  $Tr_B(\sigma)$  for eigenvalue  $\lambda$ . We have:

$$\lambda u_A(x) = u_A(\lambda x) = u_A(Tr_B(\sigma)(x)) = Tr_B(\sigma)(u_A(x)).$$

We see that  $u_A(x)$  is an eigenvector of  $Tr_B(\sigma)$  for eigenvalue  $\lambda$ . Since  $\lambda$  is a non-degenerate eigenvalue of  $Tr_B(\sigma)$ ,  $u_A(x)$  is colinear with  $x$  and  $u_A(x) = e^{i\varphi}x$  for some  $\varphi \in [0, 2\pi[$ . We see that, in a basis of eigenvectors of  $Tr_B(\sigma)$ , the unitary operation  $u_A$  is represented by a diagonal matrix (whose diagonal entries all have modulus 1). Similarly for  $u_B$  in a basis of eigenvectors of  $Tr_A(\sigma)$ . We conclude that, in the basis whose elements are the tensor products of the bases of  $A$  and  $B$  just considered, the unitary  $u_A \otimes u_B$  is represented by a diagonal matrix (whose diagonal entries have modulus 1). The conclusion of the theorem follows. ■

### 3.7 Mixed state equivalence for 2 qubits

We can now give an example of how the generalization of the *if* part of Nielsen's result to mixed states fail, even for 2-entanglement, as announced in Section 3.1.

**Theorem 13** Assume  $\lambda_1^0 + \lambda_2^0 + \lambda_3^0 \leq 2$  and  $\lambda_3^0 > 1/2$ . Let  $y$  and  $z$  be the solutions of Theorem 11, which both exist under the assumptions. Let  $\sigma$  be the two-qubits mixed state defined by  $\sigma = Tr_{\mathbf{Q}_3}(p_y)$  and let  $\tau = Tr_{\mathbf{Q}_3}(p_z)$ . The spectra of  $\sigma$  and  $\tau$  are equal, their partial traces are the same:  $Tr_{\mathbf{Q}_2}(\sigma) = Tr_{\mathbf{Q}_2}(\tau)$  and  $Tr_{\mathbf{Q}_1}(\sigma) = Tr_{\mathbf{Q}_1}(\tau)$ , but  $\sigma$  and  $\tau$  are not equivalent under local unitary transformations.

**Proof:** By properties of 2-entanglement, the spectrum of  $\sigma$  is equal (with suitable padding with zeros) to the spectrum of  $Tr_{\mathbf{Q}_1 \otimes \mathbf{Q}_2}(p_y)$ . The spectrum of  $\tau$  is equal to the spectrum of  $Tr_{\mathbf{Q}_1 \otimes \mathbf{Q}_2}(p_z)$  and those traces are equal by construction. By Equation 9,  $Tr_{\mathbf{Q}_2}(\sigma) = Tr_{\mathbf{Q}_2 \otimes \mathbf{Q}_3}(p_y) = Tr_{\mathbf{Q}_2 \otimes \mathbf{Q}_3}(p_z) = Tr_{\mathbf{Q}_2}(\tau)$  and similarly for  $Tr_{\mathbf{Q}_1}(\sigma)$ . We are left to prove that  $\sigma$  and  $\tau$  are not equivalent under local unitary transformations. We shall use the contrapositive of Theorem 12. The conclusion of the theorem does not hold since, for example, we have  $\sigma_{1,1} = y_{000}y_{000}^* + y_{001}y_{001}^* = (\lambda_1^0 + \lambda_2^0 + \lambda_3^0 - 1) / 2$  and  $\tau_{1,1} = z_{000}z_{000}^* + z_{001}z_{001}^* = (\lambda_1^0 + \lambda_2^0 - \lambda_3^0) / 2$ . Both are positive real numbers and they are different since  $\lambda_3^0 > 1/2$ . But the first assumption holds: note no eigenvalue is degenerate since  $\lambda_3^0 > 1/2$  implies  $\lambda_i^0 > \lambda_i^1$  for every  $i$ . Since  $x$  and  $y$  are solutions of the equations we know that  $\rho_i^y = \rho_i^x$  (for every  $i$ ). We conclude that the second hypothesis does not hold. ■

### 3.8 Pure state equivalence under local unitary operations

We are now interested in studying whether any two solutions of Equations (34) to (36) above are equivalent under local unitary operations.

Two solutions that differ only by the phase factors  $\theta$  are equivalent but, at least in the generic situation described in Theorem 13, the solutions  $y$  and  $z$  are not equivalent.

For 2-party entanglement, pure global states  $x$  and  $y$  of  $A \otimes B$  are equivalent under local unitary operations iff the mixed states  $\rho_A^x = \text{Tr}_B(p_x)$  and  $\rho_A^y = \text{Tr}_B(p_y)$  have the same spectrum ( $p_z$  is the projection on  $z$ ). This result cannot be generalized to 3-party entanglement.

**Theorem 14** Assume  $\lambda_1^0 + \lambda_2^0 + \lambda_3^0 \leq 2$  and  $\lambda_3^0 > 1/2$ . Let  $y$  and  $z$  be the solutions of Theorem 11, which both exist under the assumptions. They define the same spectra on each of the  $\mathbf{Q}_i$  for  $i = 1, 2, 3$  but they are not equivalent under local unitary transformations.

**Proof:** By Theorem 13. ■

The question whether pure states  $y$  and  $z$  as above can be obtained from each other with probability 1 by LOCC operations is open.

## Acknowledgements

Dorit Aharonov convinced me that entanglement was fundamental and pointed me to the right places. Discussions with her and with Michael Ben-Or are gratefully acknowledged.

## References

- [1] John S. Bell. On the Einstein Podolsky Rosen paradox. *Physics*, 1:195–200, 1964.
- [2] Ky Fan. On a theorem of Weyl concerning eigenvalues of linear transformations I. *Proceedings of the National Academy of Sciences, U.S.A.*, 35:652–655, 1949.
- [3] Albert W. Marshall and Ingram Olkin. *Inequalities: Theory of Majorization and Its Applications*, volume 143 of *Mathematics in Science and Engineering*. Academic Press, London, 1979.
- [4] M. A. Nielsen. Conditions for a class of entanglement transformations. *Physical Review Letters*, 83(2):436–439, 1999. arXiv:quant-ph/9811053.
- [5] M. B. Plenio and S. Virmani. An introduction to entanglement measures. *Quantum Information and Computation*, 7:1–51, 2007. arXiv:quant-ph/0504163.
- [6] John Preskill. Lecture notes for quantum computation course. <http://www.theory.caltech.edu/~preskill/ph229/#lecture>.

## A A theorem of Y. Fan

**Definition 6** Let  $\mathbf{H}$  be an  $n$ -dimensional Hilbert space and let  $k$  be a natural number  $1 \leq k \leq n$ . A set  $x_1, \dots, x_k$  of vectors of  $\mathbf{H}$  is said to be a  $k$ -basis iff all the vectors  $x_i$  are unit vectors and pairwise orthogonal.

The following is a slightly modified version of a result (Theorem 1) of Y. Fan [2]. The proof presented here is Fan's.

**Theorem 15 (Y. Fan , 1949)** Let  $\mathbf{H}$  be an  $n$ -dimensional Hilbert space and let  $q$  be a natural number  $1 \leq q \leq n$ . Assume  $A : \mathbf{H} \rightarrow \mathbf{H}$  is a self-adjoint linear operator and that  $Sp(A) = \lambda_1, \dots, \lambda_n$  with  $\lambda_i \geq \lambda_{i+1}$  for any  $i$ ,  $1 \leq i < n$ . For any  $q$ -basis  $b = \{x_i\}_{i=1, \dots, q}$ , define the real nonnegative quantity  $w_A(b) = \sum_{i=1}^q \langle x_i | A | x_i \rangle$ . The quantity  $w_A(b)$  is maximal (among all  $q$ -bases) if and only if  $b$  is composed of eigenvectors of  $A$  for each of the eigenvalues  $\lambda_1, \dots, \lambda_q$ . The maximal value of  $w_A(b)$  is the sum of the  $q$  largest eigenvalues of  $A$ :  $\sum_{i=1}^q \lambda_i$ .

**Proof:** Let  $\{x_i\}_{i=1, \dots, q}$  be any  $q$ -basis and let  $\{y_j\}_{j=1, \dots, n}$  be a basis for  $\mathbf{H}$  with  $y_j$  an eigenvector of  $A$  for eigenvalue  $\lambda_j$ , for  $j = 1, \dots, n$ . We have:

$$\begin{aligned} \langle x_i | A | x_i \rangle &= \sum_{j=1}^n \lambda_j |\langle y_j | x_i \rangle|^2 = \\ &= \sum_{j=1}^n (\lambda_j - \lambda_q) |\langle y_j | x_i \rangle|^2 + \lambda_q \sum_{j=1}^n |\langle y_j | x_i \rangle|^2 = \\ &= \sum_{j=1}^n (\lambda_j - \lambda_q) |\langle y_j | x_i \rangle|^2 + \lambda_q \|x_i\|^2 = \sum_{j=1}^n (\lambda_j - \lambda_q) |\langle y_j | x_i \rangle|^2 + \lambda_q \end{aligned}$$

Therefore

$$\sum_{i=1}^q \langle x_i | A | x_i \rangle = \sum_{j=1}^n (\lambda_j - \lambda_q) \sum_{i=1}^q |\langle y_j | x_i \rangle|^2 + q\lambda_q.$$

Since  $\lambda_j - \lambda_q \leq 0$  for  $j > q$  and  $\lambda_j - \lambda_q \geq 0$  for  $j \leq q$  we have:

$$\sum_{i=1}^q \langle x_i | A | x_i \rangle \leq \sum_{j=1}^q (\lambda_j - \lambda_q) \sum_{i=1}^q |\langle y_j | x_i \rangle|^2 + q\lambda_q = \sum_{j=1}^q (\lambda_j - \lambda_q) + q\lambda_q = \sum_{j=1}^q \lambda_j$$

and  $\sum_{j=1}^q \lambda_j$  is an upper bound for  $w_A(b)$ . But we have  $\sum_{i=1}^q \langle x_i | A | x_i \rangle = \sum_{j=1}^q \lambda_j$  if and only if,

- for any  $j$ ,  $q < j \leq n$  one has  $(\lambda_j - \lambda_q) \sum_{i=1}^q |\langle y_j | x_i \rangle|^2 = 0$ , i.e., for any  $j$  such that  $\lambda_j < \lambda_q$  and for any  $i$ ,  $1 \leq i \leq q$  the vectors  $y_j$  and  $x_i$  are orthogonal, and

- for any  $j$ ,  $1 \leq j \leq q$ , the vector  $y_j$  is orthogonal to all vectors  $x_i$  for  $q < i \leq n$ .

We conclude that  $w_A(b)$  is equal to  $\sum_{j=1}^q \lambda_j$  iff, for every  $q$ ,  $1 \leq q \leq n$ , the subspace spanned by  $\{x_i\}_{i=1,\dots,q}$  is the subspace spanned by  $\{y_i\}_{i=1,\dots,q}$ . ■

## B The state resulting of a measurement majorizes the initial state only in expectation

Consider a global space  $\mathbf{H} = \mathbf{H}_a \otimes \mathbf{H}_b$  where  $\mathbf{H}_a$ , Alice's system, consists of two qubits (qubits 1 and 2) and Bob's system consists of one qubit (qubit 3). Let the global state be

$$h = \sqrt{p/3} |000\rangle + \sqrt{p/3} |001\rangle + \sqrt{p/3} |111\rangle + \sqrt{(1-p)/2} |010\rangle + \sqrt{(1-p)/2} |100\rangle$$

for some  $p$ ,  $0 \leq p < 1$ . The local mixed state for Bob is

$$\begin{pmatrix} 1 - 2p/3 & p/3 \\ p/3 & 2p/3 \end{pmatrix}$$

whose eigenvalues are  $(1 \pm \sqrt{1 - 4/3 p(2 - 5/3 p)})/2$ . As expected, if  $p$  is small, the mixed state for Bob is almost the pure state  $|0\rangle$ , and one of the eigenvalues is close to 1, the other close to 0. If Alice's measures her local state by testing it on the orthogonal subspaces spanned by  $|00\rangle$  and  $|11\rangle$  on one hand and by  $|01\rangle$  and  $|10\rangle$  on the other hand and if she gets the first subspace as an answer, then the global state of the system will be:

$$h' = \sqrt{1/3}(|000\rangle + |001\rangle + |111\rangle)$$

and Bob's local mixed state will be

$$\begin{pmatrix} 1/3 & 1/3 \\ 1/3 & 2/3 \end{pmatrix}$$

whose eigenvalues are  $(1 \pm \sqrt{5/3})/2$ . If  $p < 0.2$  then  $(1 + \sqrt{5/3})/2$  is less than  $(1 + \sqrt{1 - 4/3 p(2 - 5/3 p)})/2$  and the mixed state of Bob after Alice's measurement is strictly more mixed, i.e., less pure than it was before.

## C Something probably well-known

I guess the following is well-known but I miss a precise reference.

**Theorem 16** *Let  $A$  be a finite dimensional Hilbert space and  $f : A \rightarrow A$  a linear operator. Then,  $Sp(f^* \circ f) = Sp(f \circ f^*)$ .*

**Proof:** Note, first, that both  $f^* \circ f$  and  $f \circ f^*$  are self-adjoint and therefore have  $\dim(A)$  real eigenvalues. We shall show that every eigenvalue  $\lambda$  of  $f^* \circ f$ , different from zero, is an eigenvalue of  $f \circ f^*$  with the same multiplicity. To this effect we note that if  $x \in A$  is an eigenvector of  $f^* \circ f$  for some eigenvalue  $\lambda \neq 0$ , then  $f(x)$  is an eigenvector of  $f \circ f^*$  for eigenvalue  $\lambda$ . Suppose indeed  $x$  and  $\lambda$  are as assumed, then  $f^*(f(x)) = \lambda x \neq \vec{0}$  and therefore  $f(x) \neq \vec{0}$ . But  $(f \circ f^*)(f(x)) = f((f^* \circ f)(x)) = f(\lambda x) = \lambda f(x)$ . We are left to show that the multiplicity of  $\lambda$  for  $f \circ f^*$  is at least its multiplicity for  $f^* \circ f$ . For this, we note that if  $y \in A$  is orthogonal to  $x$ , then  $f(y)$  is orthogonal to  $f(x)$ . Indeed,  $\langle f(y) | f(x) \rangle = \langle y | (f^* \circ f)(x) \rangle = \langle y | \lambda x \rangle = \lambda \langle y | x \rangle = 0$ . ■